2014/2 ENGINEERING DEPARTMENTS PHYSICS 2
RECITATION 6
(SOURCES OF THE MAGNETIC FIELD)

1. As shown in **Figure 1**, a closed loop carrying a current ***I*** consists of four parts.

a) In unit-vector notation, find the magnetic field of the closed loop at point O, using the Biot-Savart rules.

b) If the closed loop is in a uniform magnetic field of $\vec{B} = B_0(4\hat{i} + 2\hat{k})$ (B_0 is a positive constant), find the magnetic force on *ab* ve *cd* parts and torque on the loop in unit-vector notation. (Please ignore the magnetic field exerted by current of the loop)

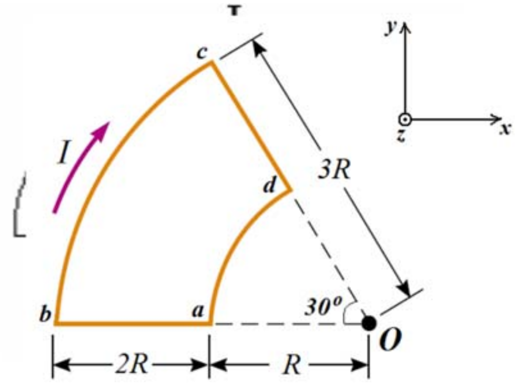
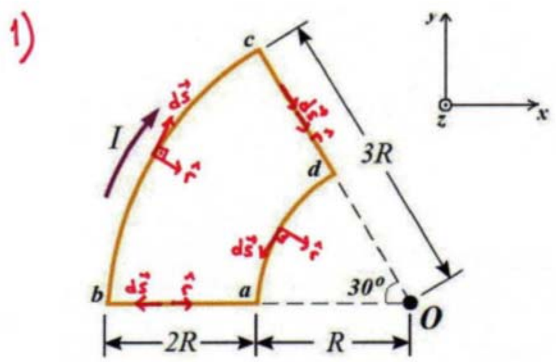


Figure 1

1)



$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{s} \times \hat{r}}{r^2}$$

$$\vec{B}_o = \vec{B}_{ab} + \vec{B}_{bc} + \vec{B}_{cd} + \vec{B}_{da}$$

$$\vec{B}_{ab} = 0 \quad (d\vec{s} \,//\, \hat{r} \ ; \ \theta = 180^\circ)$$

$$\vec{B}_{cd} = 0 \quad (d\vec{s} \,//\, \hat{r} \ ; \ \theta = 0^\circ)$$

$$\vec{B}_o = \vec{B}_{bc} + \vec{B}_{da}$$

$$\vec{B}_{bc} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{s} \times \hat{r}}{r^2}$$

$$d\vec{s} \times \hat{r} = ds(-\hat{k})$$

$$|d\vec{s} \times \hat{r}| = 3R\,d\theta$$

$$\vec{B}_{bc} = \frac{\mu_0 I}{4\pi} \int_0^{\pi/6} \frac{3R\,d\theta}{(3R)^2}(-\hat{k}) = \boxed{\frac{\mu_0 I}{72R}(-\hat{k})}$$

$$\vec{B}_{da} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{s} \times \vec{r}}{r^2}$$

$$d\vec{s} \times \hat{r} = ds(\hat{k})$$

$$|d\vec{s} \times \hat{r}| = R\,d\theta$$

$$\vec{B}_{da} = \frac{\mu_0 I}{4\pi} \int_0^{\pi/6} \frac{R\,d\theta}{R^2}(\hat{k}) = \boxed{\frac{\mu_0 I}{24R}\hat{k}}$$

$$\vec{B}_o = \frac{\mu_0 I}{72R}(-\hat{k}) + \frac{\mu_0 I}{24R}\hat{k}$$

$$\boxed{\vec{B}_o = \frac{\mu_0 I}{36R}\hat{k}}$$

b) $\vec{F}_B = I\vec{\ell} \times \vec{B}$

ab :
$$\vec{\ell} = 2R(-\hat{i})$$
$$\vec{B} = B_0(4\hat{i} + 2\hat{k})$$

$$\vec{F}_{ab} = I\,2R(-\hat{i}) \times B_0(4\hat{i} + 2\hat{k})$$

$$\boxed{\vec{F}_{ab} = 4IRB_0\,\hat{j}}$$

cd :
$$\vec{\ell} = 2R\cos 30^\circ\,\hat{i} - 2R\sin 30^\circ\,\hat{j}$$
$$\vec{B} = B_0(4\hat{i} + 2\hat{k})$$

$$\vec{F}_{cd} = I\,2R(\cos 30^\circ\,\hat{i} - \sin 30^\circ\,\hat{j}) \times B_0(4\hat{i} + 2\hat{k})$$

$$\boxed{\vec{F}_{cd} = 2IRB_0(-\hat{i} - \sqrt{3}\,\hat{j} + 2\hat{k})}$$

$$\vec{\tau} = I\vec{A} \times \vec{B}$$

$$\vec{A} = \left[\frac{\pi}{12}(3R)^2 - \frac{\pi}{12}R^2\right](-\hat{k})$$

$$\vec{B} = B_0(4\hat{i} + 2\hat{k})$$

$$\vec{\tau} = I\,\frac{2\pi}{3}R^2(-\hat{k}) \times B_0(4\hat{i} + 2\hat{k})$$

$$\boxed{\vec{\tau} = \frac{8\pi}{3}IR^2B_0\,(-\hat{j})}$$

2. In unit-vector notation, what is the magnetic field of the closed loop at point *P* as shown in **Figure 2**?

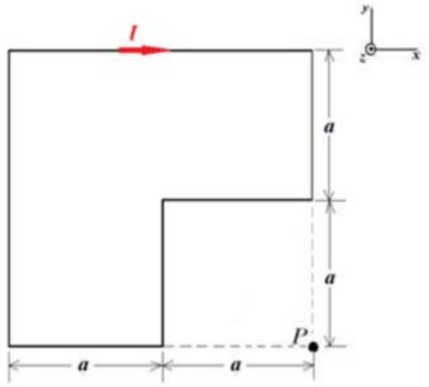
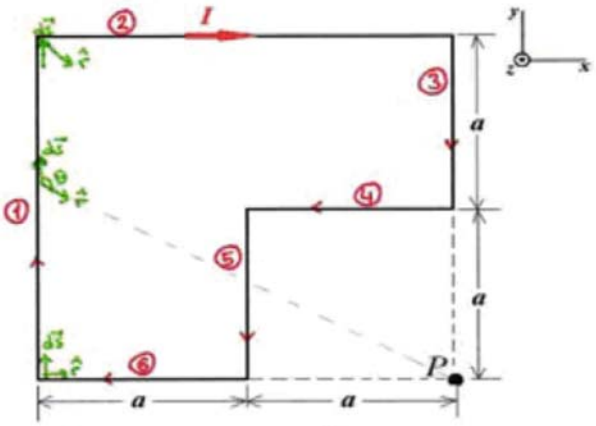


Figure 2



$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{s} \times \hat{r}}{r^2}$$

$$\vec{B}_P = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4 + \vec{B}_5 + \vec{B}_6$$

$$\vec{B}_3 = 0 \quad (d\vec{s} \parallel \hat{r} \ ; \ \theta = 0^\circ)$$

$$\vec{B}_6 = 0 \quad (d\vec{s} \parallel \hat{r} \ ; \ \theta = 180^\circ)$$

$$\vec{B}_P = \vec{B}_1 + \vec{B}_2 + \vec{B}_4 + \vec{B}_5$$

$$d\vec{s} \times \hat{r} = dy \sin\theta \, (-\hat{k})$$

$$\vec{B}_1 = \frac{\mu_0 I}{4\pi} \int \frac{dy \sin\theta}{r^2} (-\hat{k})$$

$$\vec{B}_1 = \frac{\mu_0 I}{4\pi} \int \frac{2a \csc^2\theta \, d\theta}{4a^2 \csc^2\theta} \sin\theta \, (-\hat{k})$$

$$\vec{B}_1 = \frac{\mu_0 I}{4\pi 2a} \int \sin\theta \, d\theta \, (-\hat{k})$$

$$y = -2a \cot\theta$$

$$dy = 2a \csc^2\theta \, d\theta$$

$$r = \frac{2a}{\sin\theta} = 2a \csc\theta$$

$$\vec{B}_1 = \frac{\mu_0 I}{4\pi 2a} (\cos\theta_i - \cos\theta_s) (-\hat{k})$$

$$\vec{B}_1 = \frac{\mu_0 I}{8\pi a} (\cos 90^\circ - \cos 135^\circ) (-\hat{k})$$

$$\vec{B}_2 = \frac{\mu_0 I}{8\pi a} (\cos 45^\circ - \cos 90^\circ) (-\hat{k})$$

$$\vec{B}_1 = \frac{\mu_0 I}{8\pi a} \frac{\sqrt{2}}{2} (-\hat{k})$$

$$\vec{B}_2 = \frac{\mu_0 I}{8\pi a} \frac{\sqrt{2}}{2} (-\hat{k})$$

$$\vec{B}_4 = \frac{\mu_0 I}{4\pi a} (\cos\theta_i - \cos\theta_s) (\hat{k})$$

$$\vec{B}_4 = \frac{\mu_0 I}{4\pi a} (\cos 90^\circ - \cos 135^\circ) (\hat{k})$$

$$\vec{B}_5 = \frac{\mu_0 I}{4\pi a} (\cos 45^\circ - \cos 90^\circ) (\hat{k})$$

$$\vec{B}_4 = \frac{\mu_0 I}{4\pi a} \frac{\sqrt{2}}{2} (\hat{k})$$

$$\vec{B}_5 = \frac{\mu_0 I}{4\pi a} \frac{\sqrt{2}}{2} (\hat{k})$$

$$\boxed{\vec{B}_P = \frac{\mu_0 I}{8\pi a} \sqrt{2} \, (\hat{k})}$$

3. **Figure 3** shows a cross section of a long conducting coaxial cable. The center conductor having a radius of $c = 0.5\ cm$ is surrounded by an outer conductor having an inner radius of $b = 2\ cm$ and an outer radius of $a = 4\ cm$. The current in the inner conductor is $I = 100\ A$ into the page and the current in the outer conductor is same current but its direction is out of the page.
Derive expressions for *B(r)* with radial distance r in the ranges

a) $(r < c)\ r = 0.3\ cm$,
b) $(c < r < b)\ r = 1\ cm$,
c) $(b < r < a)\ r = 3\ cm$,
d) $(r > a)\ r = 4\ cm$.

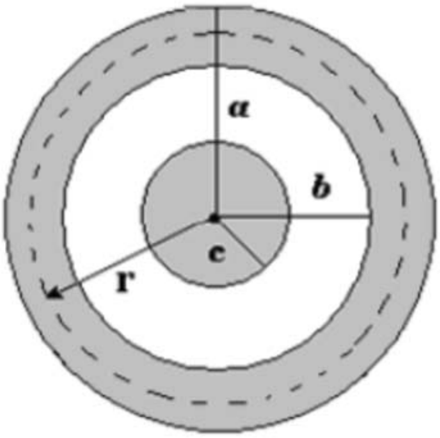
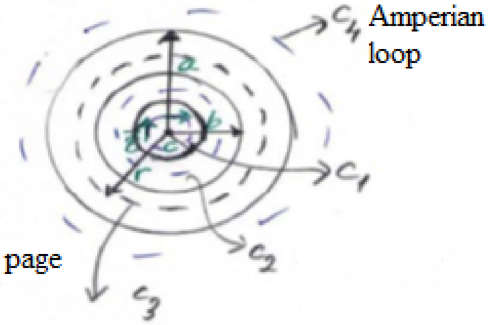


Figure 3

a) $\oint_C \vec{B}\cdot d\vec{s} = \mu_0 I_{enc}$ Ampere's law



$r < c$ $r = 0.3\ cm$, C_1 Amperian loop

$I_{enc} = JA_1$

$A_1 = \pi r^2$

$I_{enc} = \frac{I}{\pi c^2}\cdot \pi r^2 = I\frac{r^2}{c^2}$ $\otimes$ into the page

$\oint_{C_1} \vec{B}_1\cdot d\vec{s} = \mu_0 I_{enc}$ $B_1(2\pi r) = \mu_0 I \frac{r^2}{c^2}$

$B_1 = \frac{\mu_0 I r}{2\pi c^2}$ at $r = 0.3\ cm$

$B_1 = \frac{4\pi\cdot 10^{-7}\cdot 100\cdot 3\cdot 10^{-3}}{2\cdot\pi\cdot 5^2\cdot 10^{-3}\cdot 10^{-3}} = 24\cdot 10^{-7}\ T$

b) $c < r < b$, $r = 1\ cm$ $\Rightarrow$ $I_{enc} = I \otimes$

$\oint_{C_2} \vec{B}_2\cdot d\vec{s} = \mu_0 I_{enc}$ $B_2 = \frac{\mu_0 I}{2\pi r}$ $B_2\Big|_{r=1cm} = \frac{4\pi\cdot 10^{-7}\cdot 100}{2\pi\cdot 1\cdot 10^{-2}} = 2\cdot 10^{-3}\ T$

c) $b < r < a$, $r = 3\ cm$ $\Rightarrow$ $I_{enc} = I - \frac{I}{\pi(a^2-b^2)}\cdot\pi(r^2-b^2)$

$\oint_{C_3} \vec{B}_3\cdot d\vec{s} = \mu_0 I_{enc}$ $(2\pi r)B_3 = I\frac{(a^2-b^2-r^2+b^2)}{a^2-b^2}\mu_0$ $B_3 = \frac{\mu_0 I(a^2-r^2)}{2\pi r(a^2-b^2)}$

$B_3\Big|_{r=3cm} = \frac{4\pi\cdot 10^{-7}\cdot 100\,\overbrace{(4^2-3^2)}^{5}\cdot 10^{-4}}{2\pi\ 3\cdot 10^{-2}\,\underbrace{(4^2-2^2)}_{12}\cdot 10^{-4}} \cong 2{,}78\cdot 10^{-4}\ T$

d) $r > a$ $\Rightarrow$ $\oint_{C_4} \vec{B}_4\cdot d\vec{s} = I_{enc}$ $I_{top} = I - I = 0$

$B_4 = 0$

$r = 4cm = a \Rightarrow B = 0$

4. As shown in **Figure 4**, two infinitively long, parallel conductors are separated by *4m*. *Wire 1* carries a current of *8 A* out of the page and *Wire 2* carries a current of *12 A* into the page. In unit-vector notation, what is the magnitude of the resulting magnetic field at point *P*? ($\mu_0 = 4\pi.10^{-7} Wb/A.m$)

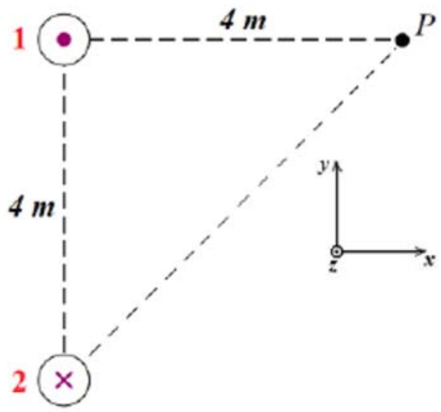
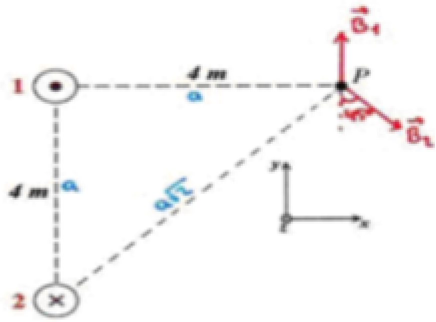


Figure 4

Ampere's Law

$$\oint \vec{B}.d\vec{s} = \mu_0 I$$



Wire 1:

$$B_1.(2\pi a) = \mu_0 I_1$$

$$B_1 = \frac{\mu_0 I_1}{2\pi a} = \frac{4\pi.10^{-7}.8}{2\pi.4} = 4.10^{-7}\ (T)$$

$$\vec{B}_1 = 4.10^{-7}\hat{j}\ (T)$$

Wire 2 :

$$B_2.(2\pi\sqrt{2}a) = \mu_0 I_2$$

$$B_2 = \frac{\mu_0 I_2}{2\sqrt{2}\pi a} = \frac{4\pi.10^{-7}.12}{2\sqrt{2}\pi.4} \cong 4,2.10^{-7}\ (T)$$

$$\vec{B}_2 = B_2 \sin 45° \hat{i} - B_2 \cos 45° \hat{j}$$

$$\vec{B}_2 = 3.10^{-7}(\hat{i} - \hat{j})\ (T)$$

$$\vec{B}_P = \vec{B}_1 + \vec{B}_2$$

$$\vec{B}_P = 4.10^{-7}\hat{j} + 3.10^{-7}(\hat{i} - \hat{j})$$

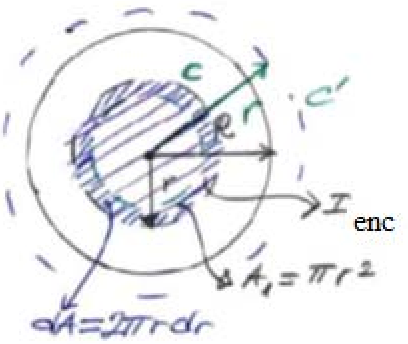
$$\boxed{\vec{B}_P = 10^{-7}(3\hat{i} + \hat{j})\ (T)}$$

5. Imagine a long, cylindrical wire of radius R that has a current density $J(r) = J_0(1 - r^2/R^2)$ for $r \le R$ and $J(r) = 0$ for $r > R$, where r is the distance from the axis of the wire.

a) Find the resulting magnetic field inside ($r \le R$)and outside ($r > R$) the wire.

b) Find the location where the magnitude of the magnetic field is a maximum, and the value of that maximum field.

a) $J(r) = J_0\left(1 - \frac{r^2}{R^2}\right) \qquad r \le R$



$r < R, \quad \oint_C \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$

$I_{enc} = \int_0^r J\,dA = \int_0^r J_0\left(1 - \frac{r^2}{R^2}\right) \cdot 2\pi r\,dr$

$I_{enc} = 2\pi J_0 \int_0^r r\,dr - \frac{2\pi J_0}{R^2} \int_0^r r^3\,dr$

$I_{enc} = \pi J_0 r^2 - \frac{\pi J_0 r^4}{2R^2} = \pi J_0\left(r^2 - \frac{r^4}{2R^2}\right)$

$B(2\pi r) = \mu_0 \pi J_0 \left(r^2 - \frac{r^4}{2R^2}\right) \qquad B = \mu_0 J_0 \left(\frac{r}{2} - \frac{r^3}{4R^2}\right)$

at $r = R \Rightarrow B = \frac{\mu_0 J_0 R}{4}$

at $r > R \Rightarrow J(r) = 0 \qquad I_{enc} = \int_0^R J\,dA + \cancel{\int_R^r J\,dA}^{\,0} = 2\pi J_0 \int_0^R r\,dr - \frac{2\pi J_0}{R^2} \int_0^R r^3\,dr$

$I_{enc} = \pi J_0\left(R^2 - \frac{R^2}{2}\right) = \pi J_0 \frac{R^2}{2}$

$\oint_{C'} \vec{B} \cdot d\vec{s} = \mu_0 I_{enc} \qquad B(2\pi r) = \pi J_0 \frac{R^2}{2} \cdot \mu_0$

$B = \frac{\mu_0 J_0 R^2}{4r}, \quad r > R$

b) at $r \le R$ in order to make B_{max}, it should be $\frac{dB}{dr} = 0$

$\frac{dB}{dr} = \frac{\mu_0 J_0}{2} - \frac{3}{4}\frac{r^2}{R^2}\mu_0 J_0 = 0 \qquad 2R^2 = 3r^2$

$r^2 = \frac{2}{3}R^2$

$r = \sqrt{2/3}\,R$

$B_{max} = \mu_0 J_0 R\left[\frac{1}{2}\left(\frac{2}{3}\right)^{1/2} - \frac{1}{4}\left(\frac{2}{3}\right)^{3/2}\right] = 0{,}272\, \mu_0 J_0 R$

6. In unit-vector notation, find the net magnetic force of an infinitely long wire carrying current ***I*** on the closed loop which is a square with the edge length ***d*** (as shown in **Figure 5**).

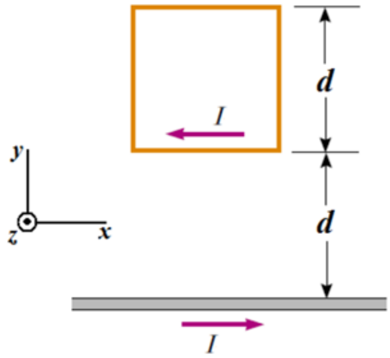
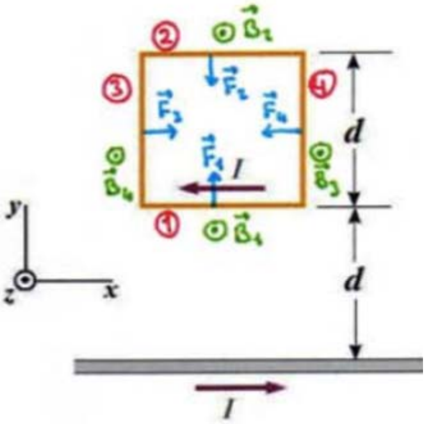


Figure 5

$\oint \vec{B}\cdot d\vec{s} = \mu_0 I$

$\vec{F}_B = I\vec{\ell}\times\vec{B}$



Wire 1

$B_1 \cdot (2\pi d) = \mu_0 I$

$\vec{B}_1 = \frac{\mu_0 I}{2\pi d}\hat{k}$

$\vec{F}_1 = I d(-\hat{i}) \times \frac{\mu_0 I}{2\pi d}\hat{k}$

$\vec{F}_1 = \frac{\mu_0 I^2}{2\pi}\hat{j}$

Wire 2

$B_2 \cdot (2\pi\, 2d) = \mu_0 I$

$\vec{B}_2 = \frac{\mu_0 I}{4\pi d}\hat{k}$

$\vec{F}_2 = I d\hat{i} \times \frac{\mu_0 I}{4\pi d}\hat{k}$

$\vec{F}_2 = \frac{\mu_0 I^2}{4\pi}(-\hat{j})$

Wire 3

$B_3 \cdot (2\pi y) = \mu_0 I$

$\vec{B}_3 = \frac{\mu_0 I}{2\pi y}\hat{k}$

$d\vec{F}_3 = I dy\hat{j} \times \vec{B}_3$

$\vec{F}_3 = \frac{\mu_0 I^2}{2\pi}\int_d^{2d}\frac{dy}{y}\hat{i}$

$\vec{F}_3 = \frac{\mu_0 I^2}{2\pi}\left[\ln y\right]_d^{2d}\hat{i}$

$\vec{F}_3 = \frac{\mu_0 I^2}{2\pi}\ln 2\,\hat{i}$

Wire 4

$\vec{F}_4 = -\vec{F}_3$

$\vec{F}_4 = \frac{\mu_0 I^2}{2\pi}\ln 2\,(-\hat{i})$

$\Sigma\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4$

$\boxed{\Sigma\vec{F} = \frac{\mu_0 I^2}{4\pi}\hat{j}}$

7. A solenoid *2.5 cm* in diameter and *30 cm* long has *300* turns and carries *12 A*.

a) Calculate the flux through the surface of a disk of radius *5 cm* that is positioned perpendicular to and centered on the axis of the solenoid, as shown in **Figure 6.a**.

b) **Figure 6.b** shows an enlarged end view of the same solenoid. Calculate the flux through the blue area, which is defined by an annulus that has an inner radius of *0.4 cm* and outer radius of *0.8 cm*.

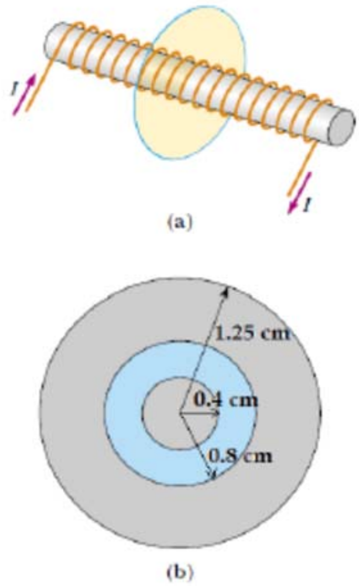


Figure 6

$N = 300$ turns

$\ell = 30\ cm = 0{,}3\ m$

$R = 1{,}25\ cm = 1{,}25 \cdot 10^{-2}\ m$

$I = 12\ A$

a) $\Phi_a = \vec{B} \cdot \vec{A} = B A$ $\quad\quad B = \mu_0 \frac{N}{\ell} I$

$\Phi_a = \left(\mu_0 \frac{N}{\ell} I\right)(\pi R^2)$

$\Phi_a = \left(4\pi \cdot 10^{-7} \cdot \frac{300}{0{,}3} \cdot 12\right) \cdot \left[\pi (1{,}25 \cdot 10^{-2})^2\right]$

$\boxed{\Phi_a \cong 7{,}4 \cdot 10^{-6}\ (Wb)}$

b) $\Phi_b = \vec{B} \cdot \vec{A} = B A$

$r_1 = 0{,}4\ cm$

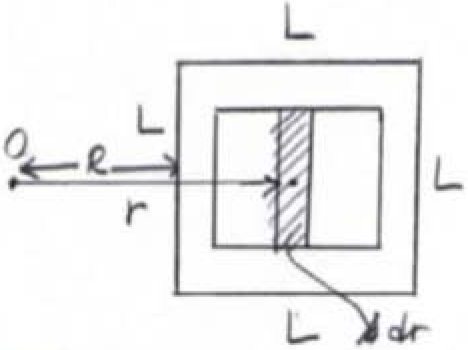
$r_2 = 0{,}8\ cm$

$\Phi_b = \left(\mu_0 \frac{N}{\ell} I\right)\left[\pi (r_2^2 - r_1^2)\right]$

$\Phi_b = \left(4\pi \cdot 10^{-7} \cdot \frac{300}{0{,}3} \cdot 12\right)\left[\pi (8^2 - 4^2) \cdot 10^{-6}\right]$

$\boxed{\Phi_b = 2{,}27 \cdot 10^{-6}\ (Wb)}$

8. Cross-section of a toroidal solenoid is a square with sides of length *L* and internal radius *R* and its shape is a cylinder. The toroid with *N* turns carries a current of I . Find an expression for the magnetic flux through the square cross-section.



the magnitude of the magnetic field at a distance r from the center of the toroid

$$B = \frac{\mu_0 NI}{2\pi r}$$

$dA = dr\,L$

$\vec{B} \parallel d\vec{A}$

the magnetic flux through a square cross-section

$$\phi_B = \int \vec{B} \cdot d\vec{A} = \frac{\mu_0 NI}{2\pi} \int_R^{R+L} L\,\frac{dr}{r} = \frac{\mu_0 NLI}{2\pi} \ln r \Big|_R^{R+L}$$

$$\phi_B = \frac{\mu_0 NIL}{2\pi} \ln\left(\frac{R+L}{R}\right)$$

9. A 5 μA current at $t = 0$ is discharging onto a capacitor having a plate area of $300\ cm^2$ and a capacitance of $10^{-7}\ F$.

a) Which ratio does the voltage between plates vary at $t = 0$?

b) Using the result of part a, calculate $d\phi_E/dt$ and magnitude of the displacement current.

a)

$$V = \frac{q}{C} \Rightarrow dV = \frac{1}{C}\,dq$$

$$\frac{dV}{dt} = \frac{1}{C}\frac{dq}{dt} = \frac{I}{C} = 5.10^{-6}(A)/2.10^{-7}(F)$$

$$= 25\ (V/s)$$

b)

$$\phi_E = E.A = q/\varepsilon_0 = CV/\varepsilon_0$$

$$\phi_E = \oint \vec{E}.d\vec{A} = \frac{q_{is}}{\varepsilon_0} = \frac{\Sigma q}{\varepsilon_0}$$

the rate of varying flux

$$\phi = \frac{CV}{\varepsilon_0} \rightarrow \frac{d\phi}{dt} = \frac{C}{\varepsilon_0}\frac{d(V)}{dt}$$

$$\boxed{\frac{d\phi}{dt} = \frac{C}{\varepsilon_0}\frac{dV}{dt} = 5,6.10^{5}\ (V.m/s)}$$

the displacement current

$$I_d = \varepsilon_0 . \frac{d\phi_E}{dt} = 8,85.10^{-12} . 5,6.10^{5} = 5.10^{-6}(A)$$