PROBLEM 1: (25 point) Two identical very long, uniform line of charge with positive linear charge density lies along x and y axes as shown in the figure.

a) Using Gauss's law, find the electric field vector $\vec{E}$ at point $P$.

$$\vec{E} \perp \mathbf{dA}_1 \text{ and } \mathbf{dA}_2$$

$$\vec{E} \parallel \mathbf{dA}_3 \text{ and } E = \text{constant on } \mathbf{dA}_3$$

$$\oint \mathbf{E} \cdot \mathbf{dA} = \frac{q_{\text{enc}}}{\varepsilon_0} \quad \text{(1)}$$

$$\oint \mathbf{E} \cdot \mathbf{dA} = \int \mathbf{E} \cdot \mathbf{dA}_1 + \int \mathbf{E} \cdot \mathbf{dA}_2 + \int \mathbf{E} \cdot \mathbf{dA}_3$$

$$= \int \mathbf{E} \cdot \mathbf{dA}_3 = EA_3 = E \cdot 2\pi r l \quad \text{(2)}$$

$$q_{\text{enc}} = \lambda l \quad \text{(2)}$$

$$E \cdot 2\pi r l = \frac{\lambda l}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi \varepsilon_0 r} \quad \text{(2)}$$

$$\vec{E}_P = \vec{E}_1 + \vec{E}_2 \quad \text{(2)}$$

$$\vec{E}_1 = \frac{\lambda}{2\pi \varepsilon_0 x} \quad \text{(2)}$$

$$\vec{E}_2 = \frac{\lambda}{2\pi \varepsilon_0 y} \quad \text{(2)}$$

$$\Delta V = \frac{\lambda}{2\pi \varepsilon_0} \ln \frac{3a}{a} \quad \text{(3)}$$

b) Find the potential difference $\Delta V = (V_B - V_A)$ between the points $A(a, a)$ and $B(a, 3a)$.

$$\Delta V = - \int \mathbf{E} \cdot \mathbf{ds} = - \left( E_x \mathbf{i} + E_y \mathbf{j} \right) \cdot (dx \mathbf{i} + dy \mathbf{j})$$

$$\Delta V = - \int_a^3 E_x dx - \int_a^{3a} E_y dy \quad \text{(3)}$$

$$= - \int_a^{3a} E_x dx - \int_a^{3a} E_y dy \quad \text{(3)}$$

$$= - \left[ \int_a^{3a} \frac{\lambda}{2\pi \varepsilon_0 y} dy \right] = - \frac{\lambda}{2\pi \varepsilon_0} \ln y \bigg|_a^{3a}$$

$$= - \frac{\lambda}{2\pi \varepsilon_0} \ln 3 \quad \text{(3)}$$

c) How much work $W$ must be done to move charge $-q$ from $A(a, a)$ to $B(a, 3a)$.

$$W = \Delta U = -q \Delta V \quad \text{(2)}$$

$$W = \frac{q \lambda}{2\pi \varepsilon_0} \ln 3 \quad \text{(3)}$$
PROBLEM 2: (25 point)

(i) Consider the circuits shown in the figure below: two coplanar, concentric rings, a small ring of radius $R_2$ and a much larger ring of radius $R_1$ with current $I_1$.

a) Using Biot-Savart's law, find the magnetic field $B_1$ at the center of rings due to current $I_1$.

\[
B = \frac{M_0 I_1}{4\pi} \int \frac{ds \sin \theta}{r^2} ds \quad \text{with} \quad r = R_1
\]

\[
B = \frac{M_0 I_1}{4\pi} \int_0^{2\pi R_1} ds \sin \theta \quad \text{with} \quad r = R_1
\]

\[
B = \frac{M_0 I_1}{2 R_1}
\]

b) Assume that the magnetic field $B_1$ is constant over the interior of the small ring and calculate the mutual inductance $M$.

\[
M = N_2 \frac{\phi_{12}}{I_1}
\]

\[
N_2 = 1 \quad \phi_{12} = \int_{\vec{B}_1 || \vec{A}_2} B_1 d\vec{A}_2
\]

\[
B_1 || A_2
\]

\[
M = \frac{M_0 I_1 \pi R_2^2}{2 R_1 I_1}
\]

(ii) A parallel plate capacitor with plates of area $A$ is filled with two dielectrics of dielectric constants $\kappa_1$ and $\kappa_2$. Each dielectric slab has thickness $d/2$. Find the stored energy in the capacitor.

\[
U = \frac{Q^2}{2C}
\]

\[
C_1 = \kappa_1 \varepsilon_0 \frac{A}{d/2}
\]

\[
C_2 = \kappa_2 \varepsilon_0 \frac{A}{d/2}
\]

\[
C_{eq} = \frac{1}{C_1} + \frac{1}{C_2}
\]

\[
C_{eq} = \frac{C_1 C_2}{C_1 + C_2}
\]

\[
U = \frac{1}{2} \frac{Q^2}{C_{eq}}
\]

\[
K_1 K_2 \left( \frac{2 \varepsilon_0 A}{d} \right)^2
\]

\[
\left( K_1 + K_2 \right) \left( \frac{2 \varepsilon_0 A}{d} \right)
\]

\[
C_0 = \frac{\varepsilon_0 A}{d}
\]

\[
C_{eq} = \frac{2 K_1 K_2}{K_1 + K_2} C_0
\]

\[
U = \frac{1}{2} \frac{Q^2}{2 K_1 K_2 C_0}
\]

\[
U = \frac{1}{2} \frac{Q^2}{K_1 + K_2}
\]
PROBLEM 3: (25 points)

a) Find the magnetic field \( B \) at the interior of the solenoid of number of turns \( N \), length \( L \) and the radius \( R \) carrying a current \( I \). Assume that the magnetic field at the exterior of the solenoid is approximately zero.

\[
\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{\text{enc}} \quad \text{(1)}
\]

\[
B_{\text{int}} = B_{\text{enc}} = \frac{\mu_0 N I}{L} \quad \text{(2)}
\]

b) If the current \( I \) increases at constant rate with time as \( \frac{dI}{dt} = +k \), answer the following questions.

i) Find the induced electro motive force \( \mathcal{E}_L \).

\[
\mathcal{E}_L = -N \frac{d\Phi_B}{dt} = -N \int B \cdot d\mathbf{A} = BA = B \pi R^2 \quad \text{(2)}
\]

\[
\mathcal{E}_L = -N \pi R^2 \frac{dB}{dt} = -N \pi R^2 \frac{\mu_0 I}{L} \frac{dI}{dt} \quad \text{(2)}
\]

\[
\mathcal{E}_L = -kN \pi R^2 \mu_0 / L \quad \text{(2)}
\]

ii) Find the induced electric field \( \mathbf{E} \) at \( r < R \).

\[
\oint \mathbf{E} \cdot d\mathbf{s} = \mathcal{E}_L \quad \text{(1)}
\]

\[
\mathbf{E} \parallel d\mathbf{s} \quad \mathbf{E} = \text{constant on } d\mathbf{s}
\]

\[
\oint \mathbf{E} \cdot d\mathbf{s} = \oint \mathbf{E} = E_2 r \quad \text{(2)}
\]

\[
E_2 r = \frac{kN^2 \mu_0}{2L} \quad \text{(2)}
\]

\[
E = \frac{kN^2 \mu_0}{2L} \quad \text{(2)}
\]

iii) A charge \( +q \) is placed at \( r < R \) from the center. The charge is initially at rest and starts moving along circular trajectory. Find the radius of curvature \( r \) at any instant.

\[
\sum F_r = ma_r
\]

\[
qE = ma_t \quad \Rightarrow r = \frac{mv}{qE} \quad \text{(1)}
\]

\[
qE = ma_t \quad \Rightarrow v = \frac{qE}{m} t \quad \text{(1)}
\]

\[
r = \frac{mv}{qE} t = \frac{E}{B} t \quad \text{(1)}
\]
PROBLEM 4: (25 point)

(i) Consider the circuit given below:

![Circuit Diagram]

a) The switch $S_1$ is closed at $t = 0$. Find the current $I_1(t)$ in the circuit as a function of time while the switch $S_2$ is open.

\[ I_1(t) = I_0 \left( 1 - e^{-t/T} \right) \]

b) Now $S_2$ is also closed. Find the currents $I_1$, $I_2$, and $I_3$ and the charge on the capacitor at steady state (after a long time).

at steady state $I_2 = 0$

\[ I_1 = I_3 = \frac{E}{2R} \]

loop 1

\[ E = I_1 R + \frac{Q}{C} \]

\[ Q = \frac{EC}{2} \]

(ii) In a series RLC circuit $I_{rms} = 0.1 A$, $\Delta V_{rms} = 60 V$, and the current leads the voltage by $\frac{\pi}{4}$ rad.

a) Calculate the average power $P_{av}$ delivered to the circuit.

\[ P_{ave} = I_{rms}^2 R = I_{rms} \Delta V_{rms} \cos \phi \]

\[ P_{ave} = (0.1)^2 (60) \cos \left(-\frac{\pi}{4}\right) \]

\[ P_{ave} = 3 \sqrt{2} \]

b) Calculate the resistance $R$ of the circuit.

\[ \tan \phi = \frac{x_L - x_C}{R} \]

\[ \tan \phi = \frac{\Delta V_{rms}}{I_{rms}} \]

\[ \left( \frac{\Delta V_{rms}}{I_{rms}} \right)^2 = R^2 + R^2 \tan^2 \phi \]

\[ R = \frac{\Delta V_{rms}}{(1 + \tan^2 \phi)^{1/2}} = \frac{60}{30 \sqrt{2}} = \frac{2}{\sqrt{2}} \]

\[ \Delta V_c(t) = \Delta V_c \sin \left( \omega t - \frac{\pi}{2} \right) \]

\[ \Delta V_c = I_m x_C \]

\[ x_L - x_C = R \tan \phi \]

\[ x_C = x_L - R \tan \phi = 800 \sqrt{2} \Omega \]

\[ I_m = \sqrt{2} I_{rms} \]

\[ \Delta V_c = \sqrt{2} I_{rms} x_C = 160 V \]

\[ \Delta V_c(t) = 160 \sin(100t - \frac{\pi}{2}) V \]